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Maze, G ; Minder, L

Abstract: This paper presents a new family of almost identities. These are based on series that sum to elements close to either rationals or rational multiples of pi. The explanation of the phenomenon takes its roots in the theory of Mellin transforms.

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A New Family of Almost Identities

Gérard Maze
e-mail: gmaze@math.unizh.ch
Mathematics Institute
University of Zurich
CH-8057 Zurich, Switzerland

Lorenz Minder
e-mail: lorenz.minder@epfl.ch
Laboratoire de Mathématiques Algorithmiques
Ecole Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland

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1 Introduction

It is well-known that a class of "almost integers" can be found using the theory of modular functions, and a few spectacular examples are given by Ramanujan [4]. They can be generated using some amazing properties of the j-function. Some of the numbers which are close approximations of integers are $\exp(\pi\sqrt{163})$ (sometimes known as Ramanujan's constant), $\exp(\pi\sqrt{37})$ and $\exp(\pi\sqrt{58})$. These irrationals come close to an integer as follows:

$$\begin{aligned}\exp(\pi\sqrt{37}) &= 199148648 - 0.219... \cdot 10^{-4}\\ \exp(\pi\sqrt{58}) &= 24591257752 - 0.177... \cdot 10^{-6}\\ \exp(\pi\sqrt{163}) &= 262537412640768744 - 0.749... \cdot 10^{-12}\end{aligned}$$

Another surprising result comes from the average length of a segment in an isosceles right triangle with catheti of unit length. If l is this average length, then

$$l = \frac{1}{30}\left(2 + 4\sqrt{2} + (4+\sqrt{2})\sinh^{-1}(1)\right) = 0.4142933026... = (\sqrt{2}-1) - 0.8... \cdot 10^{-4}.$$

Such astonishing non-equalities are usually called almost identities or non-identities. Many examples of such unexpected behaviour are known [5]. The four examples above are however different in essence: the first three come from a deep property of a complex mathematical object (the j-function) and the last has a good chance to be a genuine arithmetical coincidence.

A natural question that comes to mind in presence of such a non-identity is therefore whether or not the phenomenon is purely coincidental, or comes from a more subtle process. For instance, in the equation

$$e^{\pi} - \pi = 19.999099979... ,$$

it is not clear at all whether the almost identity pops up from a deep connection between e and π or just because the expression *happens* to be close to 20.

Recently, J.M. Borwein and P.B. Borwein discovered several families of almost identities [2], leading to a systematic study of such phenomena. These were based on mathematical concepts that lead to clear explanations. Among the non-identities studied by these authors, let us mention the following striking example:

$$\sum_{k=-\infty}^{\infty} \frac{1}{10^{(k/100)^2}} \cong 100\sqrt{\frac{\pi}{\ln(10)}},$$

correct to at least 18,000 digits. In this situation, the almost identity is *not* a coincidence. From the same viewpoint, let us mention as well the sequence

$$h_n = \frac{n!}{2(\ln(2))^{n+1}},$$

for $1 \leqslant n \leqslant 17$, discovered by D. Hickerson. These numbers are close to integers due to the fact that the above quotient is the dominant term in an infinite series whose sum is the number of possible outcomes of a race between n people (where ties are allowed). See [5] for the exact expression of these numbers. Here, once again, no coincidence.

While we were studying the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{1+2^k x} \ , \quad x \in \,]0,1],$$

that appears in the analysis of the complexity of the binary gcd algorithm, we came to find a new family of almost identities. Let us define the real numbers u_n as follows:

$$u_n := \ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{k/2} + 2^{-k/2}\right)^n} \ , \quad n \in \mathbb{N} - \{0\}.$$

The following equalities show the very strange behaviour of the almost identities generated by the sequence $\{u_n\}$.

$$\begin{array}{rcl}
u_1 & = & \pi \ + \ 0.53... \cdot 10^{-11} \\
u_2 & = & 1 \ + \ 0.48... \cdot 10^{-10} \\
u_3 & = & \dfrac{\pi}{2^3} \ + \ 0.22... \cdot 10^{-9} \\
u_4 & = & \dfrac{1}{6} \ + \ 0.67... \cdot 10^{-9} \\
u_5 & = & \dfrac{3\pi}{2^7} \ + \ 0.15... \cdot 10^{-8} \\
u_6 & = & \dfrac{1}{30} \ + \ 0.29... \cdot 10^{-8} \\
\vdots & \vdots & \qquad\qquad \vdots
\end{array}$$

This article presents an explanation of this phenomenon and sheds light on the relation between u_n and u_{n+2}. We first study the cases with $n = 1$ and $n = 2$ by using the theory of Mellin transforms. From there, we exhibit the recurrence relation

$$u_n = \left(\frac{1}{4} \cdot \frac{n-2}{n-1}\right) u_{n-2} + r_n$$

and give the explicit values of r_n satisfying $0 < r_n \leqslant r_{10} = 0.7227399... \cdot 10^{-8}$, $\forall n \in \mathbb{N}$. We also present a generalization of the phenomenon, leading to, e.g., the almost-identity

$$\ln(4) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{2^{-k} + 2^{k}} = \pi \ + \ 0.82... \cdot 10^{-5}.$$

In this article, we will use the notation $f(x) \sim_a g(x)$ for equivalent functions in a neighbourhood of a and $\log_2 x$ for the logarithm in base 2 of x. Also, the set $\mathbb{N}$ is considered to contain the integer 0 in the sequel.

2 The cases $n = 1$ and $n = 2$

The first two cases in our list are

$$u_1 = \ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{2^{-k/2} + 2^{k/2}} = \pi \ + \ 0.53... \cdot 10^{-11}$$

and

$$u_2 = \ln(2) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{-k/2} + 2^{k/2}\right)^2} = 1 \ + \ 0.48... \cdot 10^{-10}.$$

In the next section, we will see that the expression of u_n, $n > 2$, can be explicited based on these first two almost identities. We therefore begin our study by these cases. Let us define the complex functions g_1 and g_2 as

$$g_1(x) = -2 \cdot \left(\arctan(\sqrt{x}) - \pi/2\right) \quad \text{and} \quad g_2(x) = \frac{1}{1+x} \ , \quad \Re x > 0$$

as well as the functions G_1 and G_2 defined as

$$G_n(x) = \sum_{k=1}^{\infty} g_n(2^k x) \ , \quad \Re x > 0 \quad , \quad n = 1, 2.$$

The convergence of G_1 is justified by the fact that in a neighbourhood of $+\infty$ we have

$$\arctan t - \pi/2 = -\int_t^{\infty} \frac{1}{1+v^2} dv = -\int_t^{\infty} \left(\frac{1}{v^2} - \frac{1}{v^4} + \frac{1}{v^6} + \cdots\right) dv = O(1/t).$$

The following equalities are justified because G_1 and G_2 converge uniformly on compact subsets of their domains, and therefore the derivative can be interchanged with the sum. Therefore,

$$\begin{aligned}
\lim_{m\to\infty} \frac{d}{du}\left[G_n(2^{-u})\right]\Big|_{u=m} &= \lim_{m\to\infty} \sum_{k=1}^{\infty} \frac{d}{du}\left[g_n(2^{k-u})\right]\Big|_{u=m} \\
&= \lim_{m\to\infty} \ln(2) \cdot \sum_{k=1}^{\infty} \frac{\left(2^{(k-m)}\right)^{n/2}}{\left(1+2^{(k-m)}\right)^n} \\
&= \lim_{m\to\infty} \ln(2) \cdot \sum_{k=1}^{\infty} \frac{1}{\left(2^{-(k-m)/2} + 2^{(k-m)/2}\right)^n} \\
&= u_n, \qquad (2.1)
\end{aligned}$$

where the limit is understood with $m \in \mathbb{N}$. The game plan is then to express the functions G_1 and G_2 in a completely different manner in order to compute these limits. The keystone of this process is the Mellin transform [3]. Recall that the Mellin transform of a locally Lebesgue integrable function $f(x)$ over $]0, \infty[$ is the function

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx.$$

The conditions $f(x) \sim_0 O(x^u)$ and $f(x) \sim_\infty O(x^v)$, with $u > v$ guarantee that $f^*(s)$ exists in the strip $-u < \Re s < -v$. Mellin's inversion formula [3, p.13] states that if f is continuous and $c \in]-u, -v[$, then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds,$$

and in a neighbourhood of 0, we have

$$f(x) = \sum_{\Re s_l < c} \mathrm{Res}(f^*(s) x^{-s}, s_l),$$

where the summation is over the poles s_l of the function $f^*(s)x^{-s}$ whose real part is strictly smaller than c.

Let $g(x)$ be a locally Lebesgue integrable function over $]0, \infty[$, $f(x) = \sum_{k=1}^\infty g(2^k x)$, and suppose that the convergence is uniform in $]0, \infty[$. Then

$$\begin{aligned} f^*(s) &= \int_0^\infty \sum_{k=1}^\infty g(2^k x) x^{s-1} dx \\ &= \sum_{k=1}^\infty \int_0^\infty g(y) y^{s-1} 2^{-ks} dy \\ &= \frac{g^*(s)}{2^s - 1}. \end{aligned} \tag{2.2}$$

Proposition 1 *For $x > 0$, we have*

$$G_1(x) = -\frac{\pi}{2} - \pi \log_2(x) + \sqrt{x} S_1(x) - \sum_{k=1}^\infty \frac{\sin(2k\pi \log_2(x))}{k \cdot \cosh\left(2k\pi^2 / \ln(2)\right)}$$

where $S_1(x)$ is a power series in x, which converges in $[0, 1[$.

Proof: As announced earlier, the idea is to use Mellin transforms in a back and forth process to reveal another expression of G_1. Using (2.2), we can write

$$G_1^*(s) = \frac{g_1^*(s)}{2^s - 1}. \tag{2.3}$$

In order to compute g_1^*, recall that in a neighbourhood of $+\infty$ we have $\arctan t - \pi/2 = O(1/t)$.

So, we can perform an integration by parts, as long as $\Re s \in]0, 1/2[$:

$$\begin{aligned}
g_1^*(s) &= -2\int_0^\infty \left(\arctan(\sqrt{x}) - \pi/2\right) x^{s-1} dx \\
&= -2 \cdot \left[\left(\arctan(\sqrt{x}) - \pi/2\right) \cdot \frac{x^s}{s}\Big|_0^\infty - \frac{1}{2s}\int_0^\infty \frac{x^{s-1/2}}{1+x} dx \right] \\
&= \frac{1}{s}\int_0^\infty \frac{x^{s-1/2}}{1+x} dx \\
&= \frac{\pi}{s\cos \pi s}.
\end{aligned}$$

The last equality comes from the relation

$$\int_0^\infty \frac{x^{s-1}}{1+x} dx = \frac{\pi}{\sin \pi s}.$$

Using Mellin's inversion formula with $c = 1/4$ and (2.3), we get

$$\begin{aligned}
G_1(x) &= \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty} \left(\frac{\pi}{s\cos\pi s}\right) \frac{x^{-s}}{2^s - 1} ds \\
&= \sum_{\Re s_l < 1/4} \mathrm{Res}\left(\left(\frac{\pi}{s\cos\pi s}\right) \frac{x^{-s}}{2^s-1}, s_l \right).
\end{aligned}$$

The poles of the function $\left(\frac{\pi}{s\cos\pi s}\right)\frac{x^{-s}}{2^s-1}$ can be partitioned as follows:

i) $s = 0$ is a pole of order two,

ii) the real simple poles $-1/2 + k$, $k \in \mathbb{Z}$,

iii) the imaginary simple poles $2k\pi i/\ln(2)$, $k \in \mathbb{Z} \setminus \{0\}$.

The residues are then

$$\begin{array}{rl}
-\pi \log_2(x) - \frac{\pi}{2} & \text{at } s = 0, \\
-\frac{(-2)^{k+2}}{(1+2k)(2^{k+1}-\sqrt{2})}\sqrt{x}x^k & \text{at } s = -1/2 - k\ ,\ k \in \mathbb{N}, \\
\frac{1}{2i}\cdot\frac{\exp(-2k\pi i \log_2(x))}{k\cdot\cosh(2k\pi^2/\ln(2))} & \text{at } s = 2k\pi i/\ln(2)\ ,\ k \in \mathbb{Z}\setminus\{0\},
\end{array}$$

and the above sum becomes

$$G_1(x) = -\frac{\pi}{2} - \pi\log_2(x) + \sum_{k=0}^\infty \frac{(-2)^{k+2}}{(1+2k)(-2^{k+1}+\sqrt{2})}\sqrt{x}x^k - \sum_{k=1}^\infty \frac{\sin(2k\pi\log_2(x))}{k\cdot\cosh\left(2k\pi^2/\ln(2)\right)}$$

which proves the proposition. □

Corollary 2 $u_1 = \pi + \sum_{k=1}^\infty \frac{2\pi}{\cosh(2k\pi^2/\ln(2))}$.

Proof: Based on (2.1), we have

$$\begin{aligned} u_1 &= \lim_{m\to\infty} \frac{d}{du}\left[G_1(2^{-u})\right]\Big|_{u=m} \\ &= \pi + \lim_{u\to\infty}\left[e^{-u/2}S_1(e^{-u})\right]' + \sum_{k=1}^{\infty} \frac{2\pi}{\cosh\left(2k\pi^2/\ln(2)\right)} \end{aligned}$$

and the last limit being equal to zero, the corollary is proven. □

The case $n = 1$ is then settled since the sum on the right-hand side of the equality of Corollary 2 is in fact small:

$$u_1 - \pi = \sum_{k=1}^{\infty} \frac{2\pi}{\cosh\left(2k\pi^2/\ln(2)\right)} = 0.538914478... \cdot 10^{-11}.$$

Proposition 3 *For $x > 0$, we have*

$$G_2(x) = -\frac{1}{2} - \log_2(x) + S_2(x) - \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{\sin(2k\pi \log_2(x))}{\sinh\left(2k\pi^2/\ln(2)\right)}$$

where $S_2(x)$ is a power series in x, converging in $[0, 1[$ such that $S_2(x) = 0$.

Proof: The proof follows the same lines as in the first case. First,

$$g_2^*(s) = \int_0^{\infty} \frac{x^{s-1}}{1+x} dx = \frac{\pi}{\sin \pi s},$$

and thus, once again based on (2.2) and (2.3), we have

$$\begin{aligned} G_2(x) &= \int_{1/2-i\infty}^{1/2+i\infty} G_2^*(s) x^{-s} ds \\ &= \int_{1/2-i\infty}^{1/2+i\infty} \left(\frac{\pi}{\sin \pi s}\right) \frac{x^{-s}}{2^s - 1} ds \\ &= \sum_{\Re s_l < 1/2} \operatorname{Res}\left(\left(\frac{\pi}{\sin \pi s}\right) \frac{x^{-s}}{2^s - 1}, s_l\right). \end{aligned}$$

The poles of the function can be partitioned as follows:

i) $s = 0$ is a pole of order two,

ii) the real simple poles k, $k \in \mathbb{Z} \setminus \{0\}$,

iii) the imaginary simple poles $2k\pi i/\ln(2)$, $k \in \mathbb{Z} \setminus \{0\}$.

The residues are then

$$\begin{array}{ll} -\log_2(x) - \frac{1}{2} & \text{at } s = 0, \\ -\frac{(-2)^k}{2^k - 1} x^k & \text{at } s = -k\ ,\ k = 1, 2, 3, ...\ , \\ \frac{\pi}{i} \cdot \frac{\exp(-2k\pi i \log_2(x))}{\ln(2) \cdot \sinh(2k\pi^2/\ln(2))} & \text{at } s = 2k\pi i/\ln(2)\ ,\ k \in \mathbb{Z} \setminus \{0\}. \end{array}$$

The new expression of G_2 is therefore

$$G_2(x) = -\frac{1}{2} - \log_2(x) - \sum_{k=1}^{\infty} \frac{(-2)^k}{2^k - 1} x^k - \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{\sin(2k\pi \log_2(x))}{\sinh\left(2k\pi^2/\ln(2)\right)}.$$

This concludes the proof. $\square$

Corollary 4 $u_2 = 1 + \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{2k\pi}{\sinh(2k\pi^2/\ln(2))}$.

Proof: We use here the same trick as in Corollary 2:

$$\begin{aligned} u_2 &= \lim_{m\to\infty} \frac{d}{du}\left[G_2(2^{-u})\right]\Big|_{u=m} \\ &= 1 + \lim_{u\to\infty}\left[S_2(e^{-u})\right]' + \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{2k\pi}{\sinh\left(2k\pi^2/\ln(2)\right)} \end{aligned}$$

and the limit being equal to zero, the corollary is proven. $\square$

Once again, this shows why the number u_2 is almost an integer. Indeed the sum on the right-hand side is fairly small:

$$u_2 - 1 = \frac{2\pi}{\ln(2)} \sum_{k=1}^{\infty} \frac{2k\pi}{\sinh\left(2k\pi^2/\ln(2)\right)} = 0.4885108992... \cdot 10^{-10}.$$

3 The recurrence relation

Having found the roots of the mystery related to the non-equalities $u_1 \neq \pi$ and $u_2 \neq 1$, we would now like to extend the method used in the previous section to understand why $u_3, u_4, ...$ are so close to "good arithmetic numbers". Looking back to the cases $n = 1, 2$, we see that the functions g_1 and g_2 played a crucial role. The key was the fact that they satisfy the equalities

$$\frac{d}{du}\left[g_n(2^{k-u})\right] = \frac{\ln 2}{\left(2^{-(k-u)/2} + 2^{(k-u)/2}\right)^n}\ , \quad n = 1, 2.$$

The next lemma shows how we can extend them:

Lemma 5 *Let $n \in \mathbb{N}$, $n > 2$, and let*

$$\begin{aligned} I_{n,k} &= \int \frac{1}{\left(2^{-(k-u)/2} + 2^{(k-u)/2}\right)^n}\, du, \\ R_{n,k} &= \frac{1}{2\ln 2 \cdot (n-1)} \left(\frac{2^{(k-u)/2}}{1 + 2^{k-u}}\right)^{n-2} \left(\frac{1 - 2^{k-u}}{1 + 2^{k-u}}\right). \end{aligned}$$

Then

$$I_{n,k} = \left(\frac{1}{4} \cdot \frac{n-2}{n-1}\right) I_{n-2,k} + R_{n,k}$$

The proof is left to the reader, who can simply differentiate and check! The equality of the previous lemma can be used as follows. For $n > 2$, we have

$$\begin{aligned}
u_n &= \ln(2) \cdot \lim_{m\to\infty} \sum_{k=1}^{\infty} \frac{1}{\left(2^{-(k-m)/2} + 2^{(k-m)/2}\right)^n} \\
&= \ln(2) \cdot \lim_{m\to\infty} \sum_{k=1}^{\infty} \frac{d}{du} \left[I_{n,k}\right]_{u=m} \\
&= \ln(2) \cdot \lim_{m\to\infty} \left(\sum_{k=1}^{\infty} \frac{d}{du} \left[\left(\frac{1}{4} \cdot \frac{n-2}{n-1} \right) I_{n-2,k} + R_{n,k} \right]_{u=m} \right) \\
&= \left(\frac{1}{4} \cdot \frac{n-2}{n-1} \right) u_{n-2} + \ln(2) \cdot \lim_{m\to\infty} \left(\sum_{k=1}^{\infty} \frac{d}{du} \left[R_{n,k}\right]_{u=m} \right). \qquad (3.1)
\end{aligned}$$

Let us define, for $n > 2$,

$$f_n(x) = \left(\frac{\sqrt{x}}{1+x} \right)^{n-2} \frac{1-x}{1+x}, \quad \text{so that} \quad \frac{1}{2\ln 2 \cdot (n-1)} f_n(2^{k-u}) = R_{n,k}.$$

If

$$F_n(x) = \sum_{k=1}^{\infty} f_n(2^k x),$$

since this function converges uniformly and absolutely on compact subsets of $\Re x > 0$, we can interchange derivation and summation to obtain

$$\frac{1}{2 \cdot (n-1)} \cdot \lim_{m\to\infty} \frac{d}{du} \left[F_n(2^{-u})\right]_{u=m} = \ln(2) \cdot \lim_{m\to\infty} \left(\sum_{k=1}^{\infty} \frac{d}{du} \left[R_{n,k}\right]_{u=m} \right). \qquad (3.2)$$

Once again, we use Mellin transforms to find another expression for each of the functions $F_n(x)$ in order to compute these limits.

Proposition 6 *The function F_n, $n > 2$, can be represented as*

$$F_n(x) = \begin{cases} S_n(x) - \dfrac{4\pi}{\ln 2} \displaystyle\sum_{k=1}^{\infty} c_k \frac{\sin(2k\pi \log_2(x))}{\sinh(2k\pi^2/\ln 2)} & \textit{when } n \textit{ is even}, \\ \sqrt{x} S_n(x) - \dfrac{4\pi}{\ln 2} \displaystyle\sum_{k=1}^{\infty} b_k \frac{\sin(2k\pi \log_2(x))}{\cosh(2k\pi^2/\ln 2)} & \textit{when } n \textit{ is odd}, \end{cases}$$

where $S_n(x)$ is a power series converging in $[0,1[$ such that $S(0) = 0$. The coefficients c_k and b_k are given by

$$\begin{aligned}
c_k &= \frac{\prod_{j=0}^{l-2} (j^2 + 4\pi^2 k^2/\ln(2)^2)}{(2l-2)!} & \textit{when } n = 2l,\ l > 1, \\
b_k &= \frac{2\pi k \prod_{j=0}^{l-2} ((j+1/2)^2 + 4\pi^2 k^2/\ln(2)^2)}{\ln(2)(2l-1)!} & \textit{when } n = 2l+1,\ l > 1.
\end{aligned}$$

Proof: First,

$$\begin{aligned} f_n^*(s) &= \int_0^\infty f_n(x)x^{s-1}dx \\ &= \int_0^\infty \frac{x^{n/2+s-2}}{(1+x)^{n-2}}\frac{1-x}{1+x}dx \\ &= \int_0^\infty \frac{x^{n/2+s-2}}{(1+x)^{n-1}}dx - \int_0^\infty \frac{x^{n/2+s-1}}{(1+x)^{n-1}}dx. \end{aligned}$$

This expression can be evaluated with the help of the Gamma function Γ. Indeed, this function satisfies, see, e.g., [1, p.47],

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}dx$$

and therefore

$$\begin{aligned} f_n^*(s) &= \frac{\Gamma(n/2+s-1)\Gamma(n/2-s)}{\Gamma(n-1)} - \frac{\Gamma(n/2+s)\Gamma(n/2-s-1)}{\Gamma(n-1)} \\ &= -2s\frac{\Gamma(n/2-1+s)\Gamma(n/2-1-s)}{\Gamma(n-1)}. \end{aligned}$$

We used the equality $\Gamma(n) = (n-1)\Gamma(n-1)$ in the last step. Based on Euler's reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, see, e.g., [1, p.9], the previous equality leads to the following expressions, both correct for $\Re s \in]0, 1/2[$:

$$f_n^*(s) = \begin{cases} \dfrac{2}{(2l-2)!}\cdot\dfrac{\pi}{\sin\pi s}\displaystyle\prod_{j=0}^{l-2}(j^2-s^2) & \text{when } n = 2l,\ l > 1, \\ \dfrac{2}{(2l-1)!}\cdot\dfrac{-\pi s}{\cos\pi s}\displaystyle\prod_{j=0}^{l-2}\left((j+1/2)^2-s^2\right) & \text{when } n = 2l+1,\ l \geqslant 1. \end{cases}$$

The equality (2.2) and (2.3) lead once again to

$$\begin{aligned} F_n(x) &= \int_{1/4-i\infty}^{1/4+i\infty} F_n^*(s)x^{-s}ds \\ &= \sum_{\Re s_l < 1/4} \operatorname{Res}\left(f_n^*(s)\frac{x^{-s}}{2^s-1}, s_l\right). \end{aligned}$$

The poles of the function can be partitioned as follows:

i) the imaginary simple poles $2k\pi i/\ln(2)$, $k \in \mathbb{Z}\setminus\{0\}$,

ii) the real simple poles:

$s = j \in \mathbb{Z}$ with $|j| > l-2$ when $n = 2l$,

$s = j + 1/2 \in \mathbb{Z} + 1/2$ with $|j| > l-2$ when $n = 2l+1$.

Note that contrary to the cases we have seen so far, there are no poles with multiplicity. The real simple poles will clearly contribute to residues of the form $a_k x^k$ when n is even and $a_k\sqrt{x}x^k$ when n is odd. We do not exhibit the coefficients a_k since we will not need them. The imaginary simple poles lead to residues at $s = 2k\pi i/\ln 2$, $k \neq 0$, which are of the form

$$\mathrm{Res}\left(f_n^*(s)\frac{x^{-s}}{2^s-1}, 2k\pi i/\ln 2\right) = f_n^*(2k\pi i/\ln 2)\frac{x^{-2k\pi i/\ln 2}}{\ln(2)}.$$

The new expression of F_n is therefore

$$F_n(x) = \begin{cases} \displaystyle\sum_{k=l-1}^{\infty} a_k x^k - \frac{4\pi}{\ln 2}\sum_{k=1}^{\infty} c_k \frac{\sin(2k\pi\log_2(x))}{\sinh(2k\pi^2/\ln 2)} & \text{when } n = 2l,\ l > 1, \\ \displaystyle\sqrt{x}\sum_{k=l-1}^{\infty} a_k x^k - \frac{4\pi}{\ln 2}\sum_{k=1}^{\infty} b_k \frac{\sin(2k\pi\log_2(x))}{\cosh(2k\pi^2/\ln 2)} & \text{when } n = 2l+1,\ l \geqslant 1, \end{cases}$$

where the coefficients c_k and b_k are given in the proposition. This finishes the proof. □

Corollary 7 *The sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation*

$$u_n = \left(\frac{1}{4}\cdot\frac{n-2}{n-1}\right)u_{n-2} + r_n$$

where

$$r_n = \begin{cases} \displaystyle\frac{2\pi}{\ln(2)(n-1)}\cdot\sum_{k=1}^{\infty} c_k \frac{2k\pi}{\sinh(2k\pi^2/\ln 2)} & \textit{when } n = 2l,\ l > 1, \\ \displaystyle\frac{2\pi}{\ln(2)(n-1)}\cdot\sum_{k=1}^{\infty} b_k \frac{2k\pi}{\cosh(2k\pi^2/\ln 2)} & \textit{when } n = 2l+1,\ l \geqslant 1. \end{cases}$$

Proof: Based on (3.1), (3.2) and the previous proposition, we have

$$u_n - \left(\frac{1}{4}\cdot\frac{n-2}{n-1}\right)u_{n-2} = \frac{1}{2\cdot(n-1)}\cdot\lim_{m\to\infty}\frac{d}{du}\left[F_n(2^{-u})\right]_{u=m}.$$

The limit in the above expression annihilates the limit of the power series of F_n and the only contributing term in the limit is the sinus series of F_n. This gives the expected expression of r_n. □

The growth of the coefficients r_n is the combined effect of the increase of the values of c_k and b_k and the decrease of $(n-1)^{-1}$. As a consequence, the sequence r_n is increasing for $n \leqslant 10$ and decreasing for $n \geqslant 10$, which gives

$$0 < r_n \leqslant r_{10} = 0.7227399... \cdot 10^{-8}.$$

We end this article by the following remark. The entire theory used here to explain why the numbers u_n are so close to elements in $\mathbb{Q}\cup\pi\mathbb{Q}$ has nothing to do with the presence of 2 in the denominator of

$$\frac{1}{\left(2^{-k/2}+2^{k/2}\right)^n}.$$

One could argue that any sum of the type

$$\ln(m) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(m^{-k/2} + m^{k/2}\right)^n}$$

has the potential to lie close to $\mathbb{Q}$ or $\pi\mathbb{Q}$ depending on the parity of n. As a matter of fact, we have, for example,

$$\begin{aligned}
\ln(4) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{2^{-k} + 2^k} &= \pi + 0.82... \cdot 10^{-5}, \\
\ln(9) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{3^{-k} + 3^k} &= \pi + 0.15... \cdot 10^{-2}, \\
\ln(4) \cdot \sum_{k=-\infty}^{\infty} \frac{1}{\left(2^{-k} + 2^k\right)^2} &= 1 + 0.37... \cdot 10^{-4}.
\end{aligned}$$

Based on what has been shown in this article, we can say that the "error" term is due to the size of $\ln(m)$ (in the hyperbolic functions of r_n) and the smaller it is, the smaller the error will be. In other words, the choice $m = 2$ is the best one can do in order to maximize the resemblance with elements in $\mathbb{Q} \cup \pi\mathbb{Q}$.

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